Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 341 (2008) 764–782
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Solutions of the quasilinear elliptic problem with a critical Sobolev–Hardy exponent and a Hardy-type term ☆

Dongsheng Kang

Department of Mathematics, South-Central University for Nationalities, Wuhan 430074, PR China

Received 15 March 2007
Available online 7 November 2007
Submitted by C. Gutierrez

Abstract

In the present paper, the quasilinear elliptic problem with a critical Sobolev–Hardy exponent and a Hardy-type term is considered. By means of a variational method, the existence of nontrivial solutions to the problem is obtained.

Keywords: Quasilinear problem; Nontrivial solution; Critical Sobolev–Hardy exponents; Variational methods

1. Introduction and main results

In this paper, we are concerned with the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{u^{p^*(s)-1}}{|x|^s} + \lambda a(x) u^{q-1} & \text{in } \mathbb{R}^N, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u \geqslant 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $N \geqslant 3$, $1 < p < N$, $\lambda > 0$, $0 \leqslant s < p$, $0 \leqslant \mu < \bar{\mu}$, $1 < q < p^*(s)$, $\bar{\mu} := (N-p)^p/p^p$ is the best Hardy constant, $p^*(s) := p(N-s)/(N-p)$ is the critical Sobolev–Hardy exponent for the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*(s)}(\mathbb{R}^N, |x|^{-s})$ and $p^*(0) = p^* := Np/(N-p)$ is the critical Sobolev exponent. $D^{1,p}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\nabla \cdot|^p\, dx)^{1/p}$, $L^{p^*(s)}(\mathbb{R}^N, |x|^{-s})$ is the weighted Sobolev space. Throughout this paper we assume that

$$(\mathcal{H}_1) \quad a(x) \in C(\mathbb{R}^N, \mathbb{R}), \quad a(x) \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N) \quad \text{and} \quad a(0) > 0.$$

☆ This work is supported partly by the National Natural Science Foundation of China (10771219) and the Science Foundation of the State Ethnic Affairs Commission of PRC (07ZN03).

E-mail address: dongshengkang@yahoo.com.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.10.058

By a solution to problem (1.1), we mean a function $u \in D^{1,p}(\mathbb{R}^N)$ and $u \geqslant 0$ in $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla v - \mu \frac{u^{p-1} v}{|x|^p} \right) dx = \int_{\mathbb{R}^N} \left(\frac{u^{p^*(s)-1} v}{|x|^s} + \lambda a(x) u^{q-1} v \right) dx \quad \text{for all } v \in D^{1,p}(\mathbb{R}^N).$$

In this paper, we employ the following norm of $D^{1,p}(\mathbb{R}^N)$,

$$\|u\| := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx \right)^{\frac{1}{p}}.$$

By Hardy inequality (see [4,9,10,13])

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leqslant \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall u \in D^{1,p}(\mathbb{R}^N),$$

so this norm is equivalent to $(\int_{\mathbb{R}^N} |\nabla u|^p dx)^{1/p}$, the usual norm in $D^{1,p}(\mathbb{R}^N)$.

For $0 \leqslant s < p$ and $1 < p < N$, the following inequality (see [4,10]) is also used in this paper,

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(s)}}{|x|^s} dx \right)^{\frac{p}{p^*(s)}} \leqslant C \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall u \in D^{1,p}(\mathbb{R}^N),$$

where C is a positive constant. If $s = 0$, it is Sobolev inequality and in the case $s = p$, it becomes Hardy inequality. Thus we call it Sobolev–Hardy inequality, which is essentially due to Caffarelli, Kohn and Nirenberg [4]. We recall that the best Hardy constant $\bar{\mu}$ was investigated in [9] and [13], the best constant in Sobolev–Hardy inequality and the extremal functions achieving the best constant were studied in [10].

By Hardy inequality and Sobolev–Hardy inequality, the following best Sobolev–Hardy constant is well defined for $1 < p < N$, $0 \leqslant s < p$ and $0 \leqslant \mu < \bar{\mu}$:

$$A_{\mu,s} := \inf_{u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p - \mu \frac{|u|^p}{|x|^p}) dx}{(\int_{\mathbb{R}^N} \frac{|u|^{p^*(s)}}{|x|^s} dx)^{\frac{p}{p^*(s)}}}.$$

In particular, $A_{0,0}$ is nothing but the well-known best Sobolev constant (see [22]).

The quasilinear problems related to Hardy inequality and Sobolev–Hardy inequality had been studied by some authors, either in the bounded domain or in the whole space $\mathbb{R}^N$, see for example [1,9,10,12,15] and references therein. Here we recall the recent important work by Abdellaoui, Felli and Peral (see [1]), where the authors studied the extremal functions which achieves the best constant $A_{\mu,0}$, investigated the properties of the extremal functions. The results can be employed in the study of problems with critical Sobolev exponent and Hardy term, see [1] and [12] for applications. We also mention that very recently, the author in [15] investigated the extremal functions by which the best constant $A_{\mu,s}$ is achieved, see Lemma 2.1 of this paper, these results are crucial for the study of problem (1.1). The method employed in [15] is a direct extension of the argument in [1].

On the other hand, it should be mentioned that, in recent years people had paid much attention to the singular semilinear problems involving Hardy inequality and Sobolev–Hardy inequality, many results were obtained, which give us very good insight to the singular semilinear problems, see for example [5–9,14,16,17] and references therein. However, compared with the semilinear case, the results for singular quasilinear equations are less, many challenging quasilinear problems involing Hardy inequality and Sobolev–Hardy inequality remain unknown and need to be investigated further.

The regular quasilinear problems without singular term had been studied extensively. Here we recall a result related to problem (1.1). Silva and Soares [20] studied the following quasilinear problem with critical Sobolev exponent:

$$\begin{cases} -\Delta_p u = u^{p^*-1} + \lambda a(x) u^{q-1} & \text{in } \mathbb{R}^N, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u \geqslant 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $a(x)$ and q satisfies some technical conditions. The existence of nontrivial solutions to problem (1.2) was obtained. Furthermore, a more general case of (1.2) was also investigated by variational methods. For related problems to (1.2), we also mention [11] and the references in [11] and [20].

The relations between (1.1) and (1.2) are obvious. If $s = 0$ and $\mu = 0$, then (1.1) becomes (1.2), which means that problem (1.1) is in fact the continuation of (1.2). As mentioned above, we know little about (1.1) and it remains meaningful for us to investigate the problem deeply. However, due to the singularities caused by the terms $1/|x|^p$ and $1/|x|^s$, problem (1.1) becomes more complicated to deal with and we have to face more difficulties.

Inspired by [15] and [20], we continue to study the nontrivial solutions to problem (1.1) in this paper. The methods we employed here are the mountain pass arguments and analysis techniques. The main results we obtained are presented in the following theorems. Our results for (1.1) are new in the singular cases, where $0 < s < p$ and $0 < \mu < \bar{\mu}$. It is easy to verify that the intervals used in the theorems for parameters μ and q are meaningful.

Theorem 1.1. *Suppose $0 \leqslant s < p$ and $(\mathcal{H}_1)$. Assume that one of the following conditions holds*:

(i) *$N \geqslant 3$ and $1 < q < \min\{p, \frac{N}{b(\mu)}\}$.*
(ii) *$N \geqslant p^2$, $q = p$ and $0 \leqslant \mu \leqslant (N - p^2)N^{p-1}p^{-p}$.*
(iii) *$N > p^2$, $0 \leqslant \mu \leqslant (N - p^2)N^{p-1}p^{-p}$ and $\tilde{q} < q < p$, where*

$$\tilde{q} := \max\left\{1, \frac{N}{b(\mu)}, \frac{p(2N - p - pb(\mu))}{N - p}\right\}.$$

Then there exists some $\lambda^* > 0$, such that problem (1.1) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$, where λ^* depends on μ, s, N, q and $\|a(x)\|_{L^{2^*/(2^*-q)}}$.

Theorem 1.2. *Assume that $a(x)$ satisfies $(\mathcal{H}_1)$, $N \geqslant 3$, $0 \leqslant \mu < \bar{\mu}$, $\lambda > 0$ and $\bar{q} < q < p^*(s)$, where*

$$\bar{q} := \max\left\{p, \frac{N}{b(\mu)}, \frac{p(2N - p - pb(\mu))}{N - p}\right\}.$$

Then problem (1.1) possesses a nontrivial solution for every $\lambda > 0$.

Theorem 1.3. *Suppose $N \geqslant 3$, $0 \leqslant s < p$, $0 \leqslant \mu < \bar{\mu}$, $a(x)$ satisfies $(\mathcal{H}_1)$ and $a(x) \geqslant 0$ in $\mathbb{R}^N$. Then we have*

(i) *If $1 < q \leqslant p$, there exists some $\lambda^* > 0$, such that (1.1) possesses a positive solution for every $\lambda \in (0, \lambda^*)$.*
(ii) *If $p < q < p^*$, then (1.1) has a positive solution for every $\lambda > 0$.*

Remark 1.4. For $0 < s < p$, $p^*(s) \leqslant q < p$ and $a(x) \geqslant 0$ in $\mathbb{R}^N$, the existence of solutions to (1.1) is obtained by Theorem 1.3. However, in the case $0 < s < p$, $p^*(s) \leqslant q < p^*$ and $a(x)$ is a sign-changing function in $\mathbb{R}^N$, the arguments used for Theorem 1.2 are not applicable, the existence or nonexistence of solution to problem (1.1) is not clear.

This paper is organized as follows. Section 2 deals with some preliminary materials and technical results. Section 3 is devoted to the proofs of Theorems 1.1–1.3. Before ending this section, we explain some notations employed in this paper: $B(a, R)$ is the ball centered at $a \in \mathbb{R}^N$ with the radius $R > 0$, $u^+ = \max\{u, 0\}$, $a^+(x) = \max\{a(x), 0\}$, $a^-(x) = \min\{a(x), 0\}$, $(D^{1,p}(\mathbb{R}^N))^{-1}$ denotes the dual space of $D^{1,p}(\mathbb{R}^N)$, $O(\varepsilon^t)$ is the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leqslant C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$, a generic infinitesimal value. In the following argument, we always employ C to denote positive constant and omit dx in integral for convenience.

2. Preliminary results

In this section, we will establish several preliminary lemmas. To this end, we first recall a recent result on the extremal functions of $A_{\mu,s}$, which will play a key role in the argument of this paper.

Lemma 2.1. *(See [15].) Assume that $1 < p < N$, $0 \leqslant s < p$ and $0 \leqslant \mu < \bar{\mu}$. Then the limiting problem*

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{u^{p^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^N \backslash \{0\}, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N \backslash \{0\}, \end{cases}$$

has positive radial ground states

$$V_\varepsilon(x) := \varepsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{x}{\varepsilon}\right) = \varepsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{|x|}{\varepsilon}\right), \quad \forall \varepsilon > 0,$$

that satisfy

$$\int\limits_{\mathbb{R}^N} \left(|\nabla V_\varepsilon(x)|^p - \mu \frac{|V_\varepsilon(x)|^p}{|x|^p}\right) dx = \int\limits_{\mathbb{R}^N} \frac{|V_\varepsilon(x)|^{p^*(s)}}{|x|^s}\, dx = (A_{\mu,s})^{\frac{N-s}{p-s}}.$$

The function $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the limiting problem with

$$U_{p,\mu}(1) = \left(\frac{(N-s)(\bar{\mu}-\mu)}{N-p}\right)^{\frac{1}{p^*(s)-p}}.$$

Furthermore, $U_{p,\mu}$ have the following properties:

$$\lim_{r\to 0} r^{a(\mu)} U_{p,\mu}(r) = C_1 > 0,$$

$$\lim_{r\to +\infty} r^{b(\mu)} U_{p,\mu}(r) = C_2 > 0,$$

$$\lim_{r\to 0} r^{a(\mu)+1} \left|U'_{p,\mu}(r)\right| = C_1 a(\mu) \geqslant 0,$$

$$\lim_{r\to +\infty} r^{b(\mu)+1} \left|U'_{p,\mu}(r)\right| = C_2 b(\mu) > 0,$$

where C_1 and C_2 are positive constants depending on p and N, $a(\mu)$ and $b(\mu)$ are zeroes of the function

$$f(\tau) = (p-1)\tau^p - (N-p)\tau^{p-1} + \mu, \quad \tau \geqslant 0,\ 0 \leqslant \mu < \bar{\mu},$$

satisfying

$$0 \leqslant a(\mu) < \frac{N-p}{p} < b(\mu) < \frac{N-p}{p-1}.$$

Furthermore, there exist positive constants C_3 and C_4 such that

$$0 < C_3 \leqslant U_{p,\mu}(x)\left(|x|^{\frac{a(\mu)}{\delta}} + |x|^{\frac{b(\mu)}{\delta}}\right)^\delta \leqslant C_4, \quad \delta = \frac{N-p}{p}.$$

To proceed, we recall the following standard definition:

Given E a real Banach space, $\Phi \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$, then $\{u_n\} \subset E$ is called a $(PS)_c$ sequence associated with functional Φ if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$.

Now we are ready to state a suitable version of the mountain pass theorem by Ambrosetti–Rabinowitz (see [2]).

Lemma 2.2. *(See [2].) Assume that E is a real Banach space, $\Phi \in C^1(E, \mathbb{R})$ with $\Phi(0) = 0$, satisfying*:

(h_1) *There exist positive constants β and ρ such that* $\inf_{\|u\|=\rho} \Phi(u) \geqslant \beta$.
(h_2) *There exists $e \in E$ with $\|e\| > \rho$, such that $\Phi(e) \leqslant 0$.*

Then there exists a $(PS)_c$ sequence $\{u_n\} \subset E$ associated with Φ, where

$$c = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} \Phi(u) \geqslant \beta$$

and

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) = e\}.$$

We also recall the following known result by Ben-Naoum, Troestler and Willem, which will be employed for the energy functional.

Lemma 2.3. *(See [3].) Assume that* $1 < p < N$, $1 < q < p^*$ *and* $a(x) \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$. *Then the functional*

$$D^{1,p}(\mathbb{R}^N) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^N} a(x)|u|^q$$

is well defined and weakly continuous.

In the following discussion, to modify the nonlinearity, we choose the cut-off function $\phi \in C_0^\infty(\mathbb{R}^N)$ with $0 \leqslant \phi(x) \leqslant 1$, $\phi \equiv 1$ on the ball $B(0,1)$, and $\phi \equiv 0$ in $\mathbb{R}^N \setminus B(0,2)$. For all $n \in \mathbb{N}$, defining $\phi_n(x) := \phi(x/n)$ and $a_n(x) := \phi_n(x)a(x)$, then we will investigate the following sequence of problems deeply:

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{u^{p^*(s)-1}}{|x|^s} + \lambda a_n(x) u^{q-1} & \text{in } \mathbb{R}^N, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u \geqslant 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1$_n$}$$

The functional corresponding to $(2.1)_n$ is given by

$$I_n(u) := \frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right) - \frac{1}{p^*(s)}\int_{\mathbb{R}^N}\frac{(u^+)^{p^*(s)}}{|x|^s} - \frac{\lambda}{q}\int_{\mathbb{R}^N} a_n(x)(u^+)^q,$$

which is well defined on $D^{1,p}(\mathbb{R}^N)$ and belongs to $C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$. Furthermore,

$$\langle I_n'(u), v\rangle = \int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u \nabla v - \mu\frac{|u|^{p-2}uv}{|x|^p}\right) - \int_{\mathbb{R}^N}\left(\frac{(u^+)^{p^*(s)-1}v}{|x|^s} + \lambda a_n(x)(u^+)^{q-1}v\right)$$

for all $u, v \in D^{1,p}(\mathbb{R}^N)$.

Now we are in the position to verify the following compactness result, the proof is long but standard, which relies on the analysis techniques and variational argument.

Lemma 2.4. *Assume that* $\{u_n\}$ *is a bounded sequence in* $D^{1,p}(\mathbb{R}^N)$, $I_n'(u_n) \to 0$ *as* $n \to \infty$. *Then there exists a subsequence of* $\{u_n\}$ *converging weekly to a solution of problem* (1.1).

Proof. We need to apply a suitable form of the concentration compactness principle, refer to [18,19] for the original version. At first, we would like to clear up a technical point. When talking about measures we mean measures with finite total mass on $\mathbb{R}^N \cup \{\infty\}$. The space $\mathbb{R}^N \cup \{\infty\}$ is given the standard topology that makes it compact. This means that the measures can be identified as the dual space $C(\mathbb{R}^N \cup \{\infty\})$. For example δ_∞ is well defined and $\delta_\infty(\varphi) = \varphi(\infty)$. Then for $0 \leqslant s < p$, there exist a subsequence (still denoted by $\{u_n\}$), $u \in D^{1,p}(\mathbb{R}^N)$, $\eta, \nu, \bar{\nu}_p \in \mathcal{M}(\mathbb{R}^N \cup \{\infty\})$ (the space of bounded Radon measures on $\mathbb{R}^N \cup \{\infty\}$), an at most countable set $\mathcal{J}$ and a set of different points $\{x_j\}_{j\in\mathcal{J}} \subset \mathbb{R}^N \setminus \{0\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } D^{1,p}(\mathbb{R}^N), \tag{2.1}$$

$$u_n \to u \quad \text{strongly in } L^r_{\mathrm{loc}}(\mathbb{R}^N),\ 1 \leqslant r < p^*, \tag{2.2}$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \tag{2.3}$$

$$|\nabla u_n|^p \rightharpoonup \eta \geqslant |\nabla u|^p + \eta_{0,s}\delta_0 + \eta_{\infty,s}\delta_\infty, \quad \text{if } 0 < s < p\ (\eta_{0,s} \geqslant 0,\ \eta_{\infty,s} \geqslant 0), \tag{2.4}$$

$$|\nabla u_n|^p \rightharpoonup \eta \geqslant |\nabla u|^p + \eta_{0,0}\delta_0 + \eta_{\infty,0}\delta_\infty + \sum_{j\in\mathcal{J}}\eta_j\delta_{x_j}, \quad \text{if } s = 0$$
$$(\eta_{0,0} \geqslant 0,\ \eta_{\infty,0} \geqslant 0,\ \eta_j \geqslant 0,\ 0 < |x_j| < \infty,\ \forall j \in \mathcal{J}), \tag{2.5}$$

$$\frac{|u_n|^{p^*(s)}}{|x|^s} \rightharpoonup \nu = \frac{|u|^{p^*(s)}}{|x|^s} + \nu_{0,s}\delta_0 + \nu_{\infty,s}\delta_\infty, \quad \text{if } 0 < s < p\ (\nu_{0,s} \geqslant 0,\ \nu_{\infty,s} \geqslant 0), \tag{2.6}$$

$$|u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \nu_{0,0}\delta_0 + \nu_{\infty,0}\delta_\infty + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \quad \text{if } s=0$$
$$(\nu_{0,0} \geqslant 0,\ \nu_{\infty,0} \geqslant 0,\ \nu_j \geqslant 0,\ 0<|x_j|<\infty,\ \forall j \in \mathcal{J}), \tag{2.7}$$
$$\frac{|u_n|^p}{|x|^p} \rightharpoonup \bar{\nu}_p = \frac{|u|^p}{|x|^p} + \nu_{0,p}\delta_0 + \nu_{\infty,p}\delta_\infty \quad (\nu_{0,p} \geqslant 0,\ \nu_{\infty,p} \geqslant 0), \tag{2.8}$$

where we have used the following quantities:

$$\eta_{\infty,s} = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p, \quad 0<s<p,$$

$$\eta_{\infty,0} = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p, \quad s=0,$$

$$\nu_{\infty,s} = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{|u_n|^{p^*(s)}}{|x|^s}, \quad 0<s<p,$$

$$\nu_{\infty,0} = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*},$$

$$\nu_{\infty,p} = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{|u_n|^p}{|x|^p}.$$

If $0 \leqslant s < p$, by the definition of $A_{\mu,s}$ we infer that

$$A_{\mu,s}(\nu_{0,s})^{\frac{p}{p^*(s)}} \leqslant \eta_{0,s} - \mu\nu_{0,p}.$$

Claim 1. *The set $\mathcal{J}$ is finite and for any $j \in \mathcal{J}$, either $\nu_j = 0$ or $\nu_j \geqslant (A_{0,0})^{\frac{N}{p}}$.*

In fact, since $x_j \neq 0, \forall j \in \mathcal{J}$, we can choose $\varepsilon > 0$ small enough such that $0 \notin B(x_j,\varepsilon)$ and $B(x_i,\varepsilon) \cap B(x_j,\varepsilon) = \emptyset$ for $i \neq j$, $i, j \in \mathcal{J}$. Taking φ_j a smooth cut-off function centered at x_j such that $0 \leqslant \varphi_j \leqslant 1$, $\varphi_j = 1$ for $|x-x_j| \leqslant \frac{\varepsilon}{2}$, $\varphi_j = 0$ for $|x-x_j| \geqslant \varepsilon$ and $|\nabla\varphi_j| \leqslant \frac{4}{\varepsilon}$, then we get

$$\langle I_n'(u_n), u_n\varphi_j\rangle = \int_{\mathbb{R}^N} \left(|\nabla u_n|^p \varphi_j + u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi_j\right) - \int_{\mathbb{R}^N} \left(\mu \frac{|u_n|^p}{|x|^p}\varphi_j + \left(u_n^+\right)^{p^*}\varphi_j + \lambda a_n(x)\left(u_n^+\right)^q \varphi_j\right).$$

Furthermore, (2.1)–(2.8) imply that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \varphi_j = \int_{\mathbb{R}^N} \varphi_j \, d\eta \geqslant \int_{\mathbb{R}^N} |\nabla u|^p \varphi_j + \eta_j, \tag{2.9}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \varphi_j = \int_{\mathbb{R}^N} \varphi_j \, d\nu = \int_{\mathbb{R}^N} |u|^{p^*} \varphi_j + \nu_j, \tag{2.10}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \lim_{n\to\infty} \left| \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla\varphi_j \right| \\
&\quad \leqslant \lim_{\varepsilon\to 0} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |u_n|^p |\nabla\varphi_j|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p \right)^{\frac{p-1}{p}} \leqslant C \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^N} |u|^p |\nabla\varphi_j|^p \right)^{\frac{1}{p}} \\
&\quad \leqslant C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,\varepsilon)} |\nabla\varphi_j|^N \right)^{\frac{1}{N}} \left(\int_{B(x_j,\varepsilon)} |u|^{p^*} \right)^{\frac{1}{p^*}} \leqslant C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,\varepsilon)} |u|^{p^*} \right)^{\frac{1}{p^*}} = 0,
\end{aligned} \tag{2.11}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x|^p}\varphi_j\right| \leqslant \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{B(x_j,\varepsilon)}\frac{|u_n|^p}{(|x_j|-\varepsilon)^p}\varphi_j\right| = 0, \tag{2.12}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} a_n(x)\big(u_n^+\big)^q\varphi_j = 0. \tag{2.13}$$

From (2.9)–(2.13) it follows that

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\big\langle I_n'(u_n), u_n\varphi_j\big\rangle \geqslant \eta_j - \nu_j. \tag{2.14}$$

By Sobolev inequality we get $A_{0,0}(\nu_j)^{\frac{p}{p^*}} \leqslant \eta_j$, thus

$$\nu_j = 0 \quad \text{or} \quad \nu_j \geqslant (A_{0,0})^{\frac{N}{P}},$$

which implies that $\mathcal{J}$ is finite.

Claim 2. *We consider the concentration at* 0, ∞ *and* x_j, $j\in\mathcal{J}$.

(i) *If* $0<s<p$, *we have that*

$$|\nabla u_n|^p - \mu\frac{|u_n|^p}{|x|^p} \rightharpoonup \eta - \mu\bar{\nu}_p \geqslant |\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + (A_{\mu,s})(\nu_{0,s})^{\frac{p}{p^*(s)}}\delta_0 + (A_{\mu,s})(\nu_{\infty,s})^{\frac{p}{p^*(s)}}\delta_\infty. \tag{2.15}$$

(ii) *If* $s=0$, *we also have*

$$\begin{aligned}|\nabla u_n|^p - \mu\frac{|u_n|^p}{|x|^p} \rightharpoonup \eta - \mu\bar{\nu}_p &\geqslant |\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + (A_{\mu,0})(\nu_{0,0})^{\frac{p}{p^*}}\delta_0 \\ &\quad + (A_{\mu,0})(\nu_{\infty,0})^{\frac{p}{p^*}}\delta_\infty + \sum_{j\in\mathcal{J}}(A_{0,0})(\nu_j)^{\frac{p}{p^*}}\delta_{x_j}.\end{aligned} \tag{2.16}$$

In fact, the concentration at x_j, $j\in\mathcal{J}$ is already clear by Claim 1, we only need to discuss the concentration at ∞ and 0.

We first study the possibility of concentration at ∞. Choose $R>0$ large enough and ψ a regular function such that $0\leqslant\psi\leqslant 1$,

$$\psi(x) = \begin{cases} 1, & |x| > R+1,\\ 0, & |x| < R,\end{cases}$$

and $|\nabla\psi| \leqslant 4/R$. By the definition of $A_{\mu,s}$ we get

$$\int_{\mathbb{R}^N}\left(|\nabla(u_n\psi)|^p - \mu\frac{|u_n\psi|^p}{|x|^p}\right) \geqslant A_{\mu,s}\left(\int_{\mathbb{R}^N}\frac{|u_n\psi|^{p^*(s)}}{|x|^s}\right)^{\frac{p}{p^*(s)}}.$$

Consequently,

$$\int_{\mathbb{R}^N}|\psi\nabla u_n + u_n\nabla\psi|^p \geqslant \mu\int_{\mathbb{R}^N}\frac{|u_n\psi|^p}{|x|^p} + A_{\mu,s}\left(\int_{\mathbb{R}^N}\frac{|u_n\psi|^{p^*(s)}}{|x|^s}\right)^{\frac{p}{p^*(s)}}. \tag{2.17}$$

From (2.17) and the following elementary inequality

$$|a+b|^p \leqslant |a|^p + C\big(|a|^{p-1}|b| + |b|^p\big), \quad \forall a,b\in\mathbb{R}^N, \tag{2.18}$$

it follows that

$$\mu \int\limits_{\mathbb{R}^N} \frac{|u_n\psi|^p}{|x|^p} + A_{\mu,s}\left(\int\limits_{\mathbb{R}^N} \frac{|u_n\psi|^{p^*(s)}}{|x|^s}\right)^{\frac{p}{p^*(s)}}$$
$$\leqslant \int\limits_{\mathbb{R}^N} |\psi\nabla u_n + u_n\nabla\psi|^p \leqslant \int\limits_{\mathbb{R}^N} \psi^p|\nabla u_n|^p + C\int\limits_{\mathbb{R}^N}\left(|\psi\nabla u_n|^{p-1}|u_n\nabla\psi| + |u_n\nabla\varphi|^p\right). \tag{2.19}$$

Furthermore, from Holder inequality we have

$$\int\limits_{\mathbb{R}^N} |\psi\nabla u_n|^{p-1}|u_n\nabla\psi| \leqslant \left(\int\limits_{R<|x|<R+1} |u_n|^p|\nabla\psi|^p\right)^{\frac{1}{p}}\left(\int\limits_{R<|x|<R+1} |\nabla u_n|^p\right)^{\frac{p-1}{p}},$$

which together with the boundedness of $\{u_n\}$ in $D^{1,p}(\mathbb{R}^N)$ implies that

$$\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} |\psi\nabla u_n|^{p-1}|u_n\nabla\psi|$$
$$\leqslant C\left(\int\limits_{R<|x|<R+1} |u_n|^p|\nabla\psi|^p\right)^{\frac{1}{p}} = C\left(\int\limits_{R<|x|<R+1} |u|^p|\nabla\psi|^p\right)^{\frac{1}{p}}$$
$$\leqslant C\left(\int\limits_{R<|x|<R+1} |u|^{p^*}\right)^{\frac{p}{p^*}}\left(\int\limits_{R<|x|<R+1} |\nabla\psi|^N\right)^{\frac{p}{N}} \leqslant C\left(\int\limits_{R<|x|<R+1} |u|^{p^*}\right)^{\frac{p}{p^*}},$$

thus

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} |\psi\nabla u_n|^{p-1}|u_n\nabla\psi| \leqslant C\lim_{R\to\infty}\left(\int\limits_{R<|x|<R+1} |u|^{p^*}\right)^{\frac{p}{p^*}} = 0.$$

The same argument also yields

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} |u_n|^p|\nabla\psi|^p = 0.$$

Hence, from (2.19) we infer that

$$\eta_{\infty,s} - \mu\nu_{\infty,p} \geqslant A_{\mu,s}(\nu_{\infty,s})^{\frac{p}{p^*}}, \quad 0 \leqslant s < p. \tag{2.20}$$

Furthermore,

$$\left|\int\limits_{\mathbb{R}^N} a_n(x)\psi\left(u_n^+\right)^q\right| \leqslant \left(\int\limits_{|x|\geqslant R} \left(u_n^+\right)^{p^*}\right)^{\frac{p}{p^*}}\left(\int\limits_{|x|\geqslant R} \left|a(x)\right|^{\frac{p^*}{p^*-q}}\right)^{\frac{p^*-q}{p^*}}$$
$$\leqslant (A_{\mu,s})^{-1}\left\|u_n^+\right\|^p\left(\int\limits_{|x|\geqslant R} \left|a(x)\right|^{\frac{p^*}{p^*-q}}\right)^{\frac{p^*-q}{p^*}} \leqslant C\left(\int\limits_{|x|\geqslant R} \left|a(x)\right|^{\frac{p^*}{p^*-q}}\right)^{\frac{p^*-q}{p^*}},$$

which implies

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int\limits_{\mathbb{R}^N} a_n(x)\psi\left(u_n^+\right)^q\right| \leqslant C\lim_{R\to\infty}\left(\int\limits_{|x|\geqslant R} \left|a(x)\right|^{\frac{p^*}{p^*-q}}\right)^{\frac{p^*-q}{p^*}} = 0.$$

Therefore, from $\lim_{R\to\infty}\lim_{n\to\infty}\langle I_n'(u_n), u_n\psi\rangle = 0$ it follows that

$$\eta_{\infty,s} - \mu\nu_{\infty,p} \leqslant \nu_{\infty,s}, \quad 0 \leqslant s < p. \tag{2.21}$$

By (2.20) and (2.21) we conclude

$$\nu_{\infty,s}=0 \quad \text{or} \quad \nu_{\infty,s} \geqslant (A_{\mu,s})^{\frac{N-s}{p-s}}, \quad 0 \leqslant s < p.$$

On the other hand, Hardy inequality implies

$$0 \leqslant \bar{\mu}\nu_{\infty,p} \leqslant \eta_{\infty,s}, \qquad 0 \leqslant \left(1-\frac{\mu}{\bar{\mu}}\right)\eta_{\infty,s} \leqslant \eta_{\infty,s}-\mu\nu_{\infty,p}.$$

If $\nu_{\infty,s}=0$, from (2.21) it follows that $\eta_{\infty,s}=\nu_{\infty,p}=0$.

The same conclusion holds for the concentration at $x_0=0$, namely

$$\eta_{0,s}-\mu\nu_{0,p} \geqslant A_{\mu,s}(\nu_{0,s})^{\frac{p}{p^*(s)}}, \quad 0 \leqslant s < p$$

and

$$\nu_{0,s}=0 \quad \text{or} \quad \nu_{0,s} \geqslant (A_{\mu,s})^{\frac{N-s}{p-s}}, \quad 0 \leqslant s < p.$$

Furthermore,

$$0 \leqslant \bar{\mu}\nu_{0,p} \leqslant \eta_{0,s}, \qquad 0 \leqslant \left(1-\frac{\mu}{\bar{\mu}}\right)\eta_{0,s} \leqslant \eta_{0,s}-\mu\nu_{0,p}.$$

If $\nu_{0,s}=0$, then $\eta_{0,s}=\nu_{0,p}=0$.

Hence (2.15) and (2.16) hold and Claim 2 is verified.

Claim 3. *Assume $0 \leqslant s < p$ and $K \subset \mathbb{R}^N \setminus \{0, x_j,\ j \in \mathcal{J}\}$ is a compact set. Then as $n \to \infty$ we have*

(i) *$u_n \to u$ strongly in $L^r(K)$ for all $1 < r < p^*$;*
(ii) *$\frac{u_n^{p^*(s)}}{|x|^s} \to \frac{u^{p^*(s)}}{|x|^s}$ strongly in $L^1(K)$ for every s with $0 \leqslant s \leqslant p$.*

In fact, (i) is obvious. If $0 < s \leqslant p$, then (ii) can be verified by the facts that $0 \notin K$, these integrals are nonsingular on K, $p < p^*$, $p^*(s) < p^*$, p and $p^*(s)$ are sub-critical. If $s=0$, then $p^*(0)=p^*$ is the critical Sobolev exponent, we can multiply (2.7) by a nonnegative function $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$, $\varphi_\varepsilon \equiv 1$ in K, $K \subset \operatorname{supp}\varphi_\varepsilon \subset A_\varepsilon$ with $A_\varepsilon \subset \mathbb{R}^N \setminus \{0, x_j,\ j \in \mathcal{J}\}$ and $\varepsilon/2 \leqslant \operatorname{dist}(\partial K, \partial A_\varepsilon) \leqslant \varepsilon$. Taking $\varepsilon \to 0$, we obtain the desired result.

Claim 4. *Assume $K \subset \mathbb{R}^N \setminus \{0, x_j,\ j \in \mathcal{J}\}$ is a compact set. Then $\nabla u_n \to \nabla u$ strongly in $(L^p(K))^N$ as $n \to \infty$.*

The argument is as follows. By the fact that the function $h: \mathbb{R}^N \mapsto \mathbb{R}$, $h(x)=|x|^p$ is strictly convex for $1 < p < N$, we deduce that

$$0 \leqslant \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u).$$

Choosing $\psi \in C_0^\infty(\mathbb{R}^N \setminus \{0, x_j,\ j \in \mathcal{J}\})$ such that $\psi=1$ on K and $0 \leqslant \psi \leqslant 1$, we have

$$\begin{aligned} 0 &\leqslant \int_K \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u) \leqslant \int_{\mathbb{R}^N} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u)\psi \\ &= \int_{\mathbb{R}^N} \big(|\nabla u_n|^p\psi - |\nabla u_n|^{p-2}(\nabla u_n \nabla u)\big)\psi - \int_{\mathbb{R}^N} |\nabla u|^{p-2}\big(\nabla u \nabla(u_n-u)\big)\psi. \end{aligned} \tag{2.22}$$

From the fact that $\lim_{n\to\infty} I_n'(u_n)=0$, for $0 \leqslant s < p$ we get

$$\begin{aligned} o(1) &= \big\langle I_n'(u_n), u\psi\big\rangle \\ &= \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u\psi) - \mu\frac{|u_n|^{p-2}u_n u\psi}{|x|^p}\right) - \int_{\mathbb{R}^N} \left(\frac{(u_n^+)^{p^*(s)-1}u\psi}{|x|^s} - \lambda a_n(x)\big(u_n^+\big)^{q-1}u\psi\right). \end{aligned} \tag{2.23}$$

Furthermore, the boundedness of $(u_n\psi)$ in $D^{1,p}(\mathbb{R}^N)$ implies that

$$o(1)=\langle I_n'(u_n),u_n\psi\rangle=\int_{\mathbb{R}^N}\left(|\nabla u_n|^p\psi+|\nabla u_n|^{p-2}(\nabla u_n\nabla\psi)u_n-\mu\frac{|u_n|^p\psi}{|x|^p}\right)$$
$$-\int_{\mathbb{R}^N}\left(\frac{(u_n^+)^{p^*(s)}\psi}{|x|^s}+\lambda a_n(x)\big(u_n^+\big)^q\psi\right). \tag{2.24}$$

From (2.22)–(2.24), as $n\to\infty$ we obtain that

$$0\leqslant\int_K\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u)$$
$$\leqslant\int_{\mathbb{R}^N}\psi\left(\frac{(u_n^+)^{p^*(s)-1}}{|x|^s}+\frac{(u_n^+)^{p-1}}{|x|^p}+\lambda a_n(x)\big(u_n^+\big)^{q-1}\right)(u_n-u)+\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}(\nabla u_n\nabla\psi)(u_n-u)$$
$$+\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla(u-u_n)\psi+o(1). \tag{2.25}$$

By the boundedness of $\{u_n\}$ in $D^{1,p}(\mathbb{R}^N)$, applying Holder inequality and Sobolev–Hardy inequality on the compact set $\bar K=\operatorname{supp}\psi$, we have

$$\left|\int_{\bar K}\frac{(u_n^+)^{p^*(s)-1}(u_n-u)}{|x|^s}\right|\leqslant\left(\int_{\bar K}\frac{(u_n^+)^{p^*(s)}}{|x|^s}\right)^{\frac{p^*(s)-1}{p^*(s)}}\left(\int_{\bar K}\frac{|u_n-u|^{p^*(s)}}{|x|^s}\right)^{\frac{1}{p^*(s)}}$$
$$\leqslant C\big\|u_n^+\big\|^{p^*(s)-1}\|u_n-u\|_{L^{p^*(s)}(\bar K,|x|^{-s})}\leqslant C\|u_n-u\|_{L^{p^*(s)}(\bar K,|x|^{-s})}, \tag{2.26}$$

$$\left|\int_{\bar K}\frac{(u_n^+)^{p-1}(u_n-u)}{|x|^p}\right|\leqslant\left(\int_{\bar K}\frac{(u_n^+)^p}{|x|^p}\right)^{\frac{p-1}{p}}\left(\int_{\bar K}\frac{|u_n-u|^p}{|x|^p}\right)^{\frac1p}\leqslant C\big\|u_n^+\big\|^{p-1}\|u_n-u\|_{L^p(\bar K,|x|^{-p})}$$
$$\leqslant C\|u_n-u\|_{L^p(\bar K,|x|^{-p})} \tag{2.27}$$

and

$$\left|\int_{\bar K}a_n(x)\big(u_n^+\big)^{q-1}(u_n-u)\right|\leqslant\big\|a(x)\big\|_{L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)}\big\|u_n^+\big\|^{q-1}_{L^{p^*}(\bar K)}\|u_n-u\|_{L^{p^*}(\bar K)}$$
$$\leqslant C\|a(x)\|_{L^{\frac{p^*}{p^*-q}}(\bar K)}\|u_n^+\|^{q-1}\|u_n-u\|_{L^{p^*}(\bar K)}\leqslant C\|u_n-u\|_{L^{p^*}(\bar K)}. \tag{2.28}$$

From (2.25)–(2.28) it follows that

$$0\leqslant\int_K\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u)\leqslant C\|u_n-u\|_{L^{p^*(s)}(\bar K,|x|^{-s})}+C\|u_n-u\|_{L^p(\bar K,|x|^{-p})}$$
$$+C\|u_n-u\|_{L^{p^*}(\bar K)}+\|\nabla\psi\|_{L^\infty(\bar K)}\|u_n\|^{p-1}\|u-u_n\|_{L^p(\bar K)}$$
$$+\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)\psi+o(1). \tag{2.29}$$

Noting that $\bar K=\operatorname{supp}\psi\subset\mathbb{R}^N\setminus\{0,x_j,\ j\in\mathcal{J}\}$, from (2.1)–(2.8), (2.29) and Claim 3 we get

$$\lim_{n\to\infty}\int_K\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u)=0. \tag{2.30}$$

Now we recall the following inequality (see [21]):

$$\big\langle |x|^{p-2}y-|y|^{p-2}x, x-y\big\rangle \geqslant \begin{cases} C_p|x-y|^p, & p\geqslant 2,\\ C_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & 1<p<2,\end{cases} \tag{2.31}$$

where $x, y\in\mathbb{R}^N$, $C_p>0$ is a constant depending on p.

If $p\geqslant 2$, from (2.30) and (2.31) we have

$$\lim_{n\to\infty}\int\limits_K |\nabla u_n-\nabla u|^p=0. \tag{2.32}$$

If $1<p<2$, by (2.30) and (2.31) we also get

$$\lim_{n\to\infty} C_p\int\limits_K \frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}=0. \tag{2.33}$$

Furthermore, Holder inequality implies that

$$\begin{aligned}\int\limits_K |\nabla u_n-\nabla u|^p&=\int\limits_K \frac{|\nabla u_n-\nabla u|^p}{(|\nabla u_n|+|\nabla u|)^{p(2-p)/2}}\big(|\nabla u_n|+|\nabla u|\big)^{\frac{p(2-p)}{2}}\\ &\leqslant\bigg(\int\limits_K \frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\bigg)^{\frac{p}{2}}\bigg(\int\limits_K(|\nabla u_n|+|\nabla u|)^p\bigg)^{\frac{2-p}{2}}\leqslant C\bigg(\int\limits_K \frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\bigg)^{\frac{p}{2}},\end{aligned}$$

which together with (2.33) yields (2.32).

Thus Claim 4 is proved.

As a direct corollary of Claim 4, we get that

Claim 5. *The sequence $\{u_n\}\subset D^{1,p}(\mathbb{R}^N)$ possesses a subsequence, still denoted by $\{u_n\}$, satisfying $\nabla u_n\to\nabla u$ for almost every $x\in\mathbb{R}^N$.*

Next we verify that $u\geqslant 0$. Since

$$\|u_n^-\|^p\leqslant\|u_n^+\|^p+\|u_n^-\|^p=\|u_n\|^p,$$

the sequence (u_n^-) is bounded in $D^{1,p}(\mathbb{R}^N)$, which combined with (2.1) and (2.3) yields that there exists a subsequence, still denoted by (u_n^-), such that $u_n^-\rightharpoonup u^-$ weakly in $D^{1,p}(\mathbb{R}^N)$ and $u_n^-\to u^-$ a.e. in $\mathbb{R}^N$.

On the other hand,

$$\big\langle I_n'(u_n), u_n^-\big\rangle=\|u_n^-\|^p\to 0\quad\text{as } n\to\infty,$$

that is, $u_n^-\to 0$ in $D^{1,p}(\mathbb{R}^N)$ as $n\to\infty$. Therefore $u^-\equiv 0$ and $u\geqslant 0$ in $\mathbb{R}^N$.

For $v\in D^{1,p}(\mathbb{R}^N)$, choosing $n_0>0$ such that $\operatorname{supp} v\subset B(0,n_0)$, then from $(2.1)_n$ we have

$$a_n(x)=a(x),\quad \forall x\in\operatorname{supp} v,\ n\geqslant n_0.$$

For all $v\in D^{1,p}(\mathbb{R}^N)$, Vitali theorem and Claim 5 imply that

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\bigg(\frac{(u_n^+)^{p^*(s)-1}v}{|x|^s}+\lambda a_n(x)\big(u_n^+\big)^{q-1}v\bigg)=\int\limits_{\mathbb{R}^N}\bigg(\frac{u^{p^*(s)-1}v}{|x|^s}+\lambda a(x)u^{q-1}v\bigg)$$

and

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\bigg(|\nabla u_n|^{p-2}\nabla u_n\nabla v-\mu\frac{|u_n|^{p-2}u_nv}{|x|^p}\bigg)=\int\limits_{\mathbb{R}^N}\bigg(|\nabla u|^{p-2}\nabla u\nabla v-\mu\frac{u^{p-1}v}{|x|^p}\bigg).$$

Since $I_n'(u_n) \to 0$ as $n \to \infty$, we deduce that

$$\int\limits_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla v - \mu \frac{u^{p-1} v}{|x|^p} - \frac{u^{p^*(s)-1} v}{|x|^s} - \lambda a(x) u^{q-1} v \right) = 0$$

holds for all $v \in D^{1,p}(\mathbb{R}^N)$, which implies u is a solution to (1.1).

Thus the proof of Lemma 2.4 is completed. □

In the following discussion, we verify that the functional I_n satisfy conditions (h_1) and (h_2) of Lemma 2.2 uniformly.

Lemma 2.5. *Consider problem* $(2.1)_n$ *and the corresponding functional* I_n.

(i) *If* $1 < q \leqslant p$, *there exists some* $\lambda^* > 0$, *such that* I_n *satisfy* (h_1) *for all* $\lambda \in (0, \lambda^*)$ *and* $n \in \mathbb{N}$.
(ii) *If* $p < q < p^*$, *then* I_n *satisfy* (h_1) *for all* $\lambda \in (0, +\infty)$ *and* $n \in \mathbb{N}$.

Proof. Applying Holder inequality, Sobolev inequality and Sobolev–Hardy inequality, for all $u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}$ we have

$$\begin{aligned}
I_n(u) &= \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|u^+|^{p^*(s)}}{|x|^s} - \frac{\lambda}{q} \int\limits_{\mathbb{R}^N} a_n(x) \left|u^+\right|^q \\
&\geqslant \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|u^+|^{p^*(s)}}{|x|^s} - \frac{\lambda}{q} \int\limits_{\mathbb{R}^N} a_n^+(x) \left|u^+\right|^q \\
&\geqslant \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|u|^{p^*(s)}}{|x|^s} - \frac{\lambda}{q} \left\|a^+(x)\right\|_{L^{\frac{p^*}{(p^*-q)}}} \left(\int\limits_{\mathbb{R}^N} |u|^{p^*} \right)^{\frac{q}{p^*}} \\
&\geqslant \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)(A_{\mu,s})^{\frac{p^*(s)}{p}}} \|u\|^{p^*(s)} - \frac{\lambda}{q(A_{\mu,0})^{\frac{q}{p}}} \left\|a^+(x)\right\|_{L^{\frac{p^*}{(p^*-q)}}} \|u\|^q \\
&= \frac{1}{p}\|u\|^p - C_3\|u\|^{p^*(s)} - C_4\lambda\|u\|^q,
\end{aligned}$$

where

$$C_3 = \frac{1}{p^*(s)(A_{\mu,s})^{\frac{p^*(s)}{p}}}, \qquad C_4 = \frac{1}{q(A_{\mu,0})^{\frac{q}{p}}} \left\|a^+(x)\right\|_{L^{\frac{p^*}{(p^*-q)}}}.$$

If $1 < q \leqslant p$, then

$$I_n(u) \geqslant \|u\|^q \left(\frac{1}{p}\|u\|^{p-q} - C_3\|u\|^{p^*(s)-q} - \lambda C_4 \right),$$

which implies that there exist $\lambda^* > 0$, $\rho > 0$ and $\beta > 0$, such that

$$I_n(u) \geqslant \beta \quad \text{for } \|u\| = \rho,\ \forall\lambda \in (0, \lambda^*),\ \forall n \in \mathbb{N}.$$

If $p < q < p^*$, then

$$I_n(u) \geqslant \|u\|^p \left(\frac{1}{p} - C_3\|u\|^{p^*(s)-p} - \lambda C_4\|u\|^{q-p} \right),$$

which implies that for all $\lambda > 0$ and $n \in \mathbb{N}$, there exist $\rho > 0$ and $\beta > 0$, such that

$$I_n(u) \geqslant \beta \quad \text{for } \|u\| = \rho.$$

Thus Lemma 2.5 is proved. □

Lemma 2.6. *The functional I_n satisfies (h_2) for every $\lambda > 0$ and $n \in \mathbb{N}$.*

Proof. From $(\mathcal{H}_1)$, there exists $R > 0$ such that $a(x) > 0, \forall x \in \bar{B}(0, 2R)$. Considering the positive function $v \in D^{1,p}(\mathbb{R}^N)$ with $\operatorname{supp} v \subset B(0, 2R)$, for all $t > 0$ we get that

$$I_n(tv) = \frac{t^p}{p}\|v\|^p - \frac{t^{p^*(s)}}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|v|^{p^*(s)}}{|x|^s} - \frac{\lambda t^q}{q} \int\limits_{\mathbb{R}^N} a_n(x)|v|^q \leqslant \frac{t^p}{p}\|v\|^p - \frac{t^{p^*(s)}}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|v|^{p^*(s)}}{|x|^s}.$$

Since $1 < q < p^*$ and $p^*(s) > p$, for $t > 0$ large enough we have that $I_n(tv) < 0$ and $\|tv\| > \rho$, where ρ is given by Lemma 2.5. The proof is completed. □

In the following, we will give some estimates for the extremal function V_ε defined in Lemma 2.1. Choose $R > 0$ as in Lemma 2.6 such that $a(x) > 0$ in $B(0, 2R)$, let $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ for $|x| \leqslant R$, $\varphi(x) = 0$ for $|x| \geqslant 2R$, set $v_\varepsilon(x) = \varphi(x)V_\varepsilon(x)$. For $\varepsilon \to 0$, the behavior of v_ε has to be the same as that of V_ε, but we need precise estimates of the error terms. For $1 < p < N$, $0 \leqslant s < p$ and $1 < q < p^*$, we have the following estimates (see [15]):

$$\|v_\varepsilon\|^p = (A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big), \tag{2.34}$$

$$\int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} = (A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)-N+s}\big), \tag{2.35}$$

$$\int\limits_{\mathbb{R}^N} |v_\varepsilon|^q \geqslant \begin{cases} C\varepsilon^{N+q(1-\frac{N}{p})}, & \frac{N}{b(\mu)} < q < p^*, \\ C\varepsilon^{N+q(1-\frac{N}{p})}|\ln\varepsilon|, & q = \frac{N}{b(\mu)}, \\ C\varepsilon^{q(b(\mu)+1-\frac{N}{p})}, & 1 \leqslant q < \frac{N}{b(\mu)}. \end{cases} \tag{2.36}$$

Lemma 2.7. *Under the assumptions of either Theorem* 1.1 *or Theorem* 1.2*, for all $\lambda > 0$ and n large enough, there exists a positive constant d_λ, such that*

$$\sup_{t\geqslant 0} I_n(tv_\varepsilon) \leqslant d_\lambda < \frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}}. \tag{2.37}$$

Proof. For $t \geqslant 0$, we consider the following functions

$$g(t) = \frac{t^p}{p}\|v_\varepsilon\|^p - \frac{t^{p^*(s)}}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} - \frac{\lambda t^q}{q} \int\limits_{\mathbb{R}^N} a(x)|v_\varepsilon|^q$$

and

$$\bar{g}(t) = \frac{t^p}{p}\|v_\varepsilon\|^p - \frac{t^{p^*(s)}}{p^*(s)} \int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}.$$

Noting that for $t \geqslant 0$ and n large enough, we have $I_n(tv_\varepsilon) \leqslant g(t)$ and

$$\sup_{t\geqslant 0} I_n(tv_\varepsilon) \leqslant \sup_{t\geqslant 0} g(t). \tag{2.38}$$

From the fact that $\lim_{t\to+\infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0, we infer $\sup_{t\geqslant 0} g(t)$ must be attained at some finite $t_\varepsilon > 0$ and $g'(t_\varepsilon) = 0$. Hence

$$0 = t_\varepsilon^{p-1}\|v_\varepsilon\|^p - t_\varepsilon^{p^*(s)-1} \int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} - \lambda t_\varepsilon^{q-1} \int\limits_{B(0,2R)} a(x)|v_\varepsilon|^q \leqslant t_\varepsilon^{p-1}\|v_\varepsilon\|^p - t_\varepsilon^{p^*(s)-1} \int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s},$$

which together with (2.34) and (2.35) implies that

$$t_\varepsilon \leqslant \|v_\varepsilon\|^{\frac{p}{p^*(s)-p}} \left(\int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} \right)^{\frac{-1}{p^*(s)-p}} \leqslant C_5,$$

where $C_5 > 0$ is a constant independent of ε. Furthermore, Lemma 2.5 and (2.34)–(2.36) yield that

$$0 < \beta \leqslant g(t_\varepsilon) \leqslant \frac{t_\varepsilon^p}{p} \|v_\varepsilon\|^p = \frac{t_\varepsilon^p}{p} \big((A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big)\big),$$

thus there exists a constant $C_6 > 0$ independent of ε, such that

$$C_6 \leqslant t_\varepsilon \leqslant C_5 \tag{2.39}$$

for ε small enough. On the other hand, by the fact that

$$\sup_{t \geqslant 0} \left(\frac{t^p}{p} B_1 - \frac{t^{p^*(s)}}{p^*(s)} B_2 \right) = \frac{p-s}{p(N-s)} B_1^{\frac{N-s}{p-s}} B_2^{-\frac{N-p}{p-s}}, \quad B_1 > 0,\ B_2 > 0,$$

we get

$$\begin{aligned}
\sup_{t \geqslant 0} \bar{g}(t) &= \frac{p-s}{p(N-s)} \|v_\varepsilon\|^{\frac{p(N-s)}{p-s}} \left(\int\limits_{\mathbb{R}^N} \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} \right)^{-\frac{N-p}{p-s}} \\
&= \frac{p-s}{p(N-s)} \big((A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big)\big)^{\frac{N-s}{p-s}} \big((A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)-N+s}\big)\big)^{-\frac{N-p}{p-s}} \\
&= \frac{p-s}{p(N-s)} (A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big) + O\big(\varepsilon^{b(\mu)p^*(s)-N+s}\big) \\
&= \frac{p-s}{p(N-s)} (A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big),
\end{aligned}$$

where we have used the fact that

$$b(\mu)p + p - N < b(\mu)p^*(s) - N + s, \quad \forall s \in [0, p),\ \mu \in [0, \bar{\mu}).$$

Furthermore, $(\mathcal{H}_1)$ implies

$$\bar{a}_0 := \inf_{x \in \bar{B}(0,2R)} a(x) > 0,$$

where R is chosen as in Lemma 2.6. Thus for all $\lambda > 0$, $1 < q < p^*(s)$ and ε small we have

$$\begin{aligned}
g(t_\varepsilon) &= \bar{g}(t_\varepsilon) - \frac{\lambda}{q} t_\varepsilon^q \int\limits_{B(0,2R)} a(x)|v_\varepsilon|^q \leqslant \sup_{t \geqslant 0} \bar{g}(t) - \frac{\lambda}{q} t_\varepsilon^q \int\limits_{B(0,2R)} a(x)|v_\varepsilon|^q \\
&= \frac{p-s}{p(N-s)} (A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big) - \frac{\lambda}{q} (C_6)^q \bar{a}_0 \int\limits_{B(0,2R)} |v_\varepsilon|^q,
\end{aligned}$$

where C_6 is the constant in (2.39). Defining the constant

$$d_\lambda := \frac{p-s}{p(N-s)} (A_{\mu,s})^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big) - \frac{\lambda}{q} (C_6)^q \bar{a}_0 \int\limits_{B(0,2R)} |v_\varepsilon|^q,$$

then from (2.38) we get

$$\sup_{t \geqslant 0} I_n(t v_\varepsilon) \leqslant g(t_\varepsilon) \leqslant d_\lambda. \tag{2.40}$$

To proceed, we need to consider the following particular cases.

(i) $\bar{q} < q < p^*(s)$ with

$$\bar{q} = \max\left\{p, \frac{N}{b(\mu)}, \frac{p(2N - b(\mu)p - p)}{N - p}\right\}.$$

In this case we have

$$\int_{\mathbb{R}^N} |v_\varepsilon|^q \geqslant C\varepsilon^{N+(1-\frac{N}{p})q}$$

and

$$b(\mu)p + p - N > N + \left(1 - \frac{N}{p}\right)q,$$

which together with (2.40) yield (2.37) for $\varepsilon > 0$ small.

(ii) $q = p$ and $0 \leqslant \mu \leqslant (N - p^2)N^{p-1}p^{-p}$.

If $b(\mu) > \frac{N}{p}$, then $b(\mu)p + p - N > p$. From (2.36) and (2.40) we get

$$\int_{\mathbb{R}^N} |v_\varepsilon|^p \geqslant C\varepsilon^p$$

and (2.37) holds for ε small.
If $b(\mu) = \frac{N}{p}$, then $b(\mu)p + p - N = p$. Furthermore, (2.36) and (2.40) yield that

$$\int_{\mathbb{R}^N} |v_\varepsilon|^p \geqslant C\varepsilon^p |\ln\varepsilon|$$

and (2.37) holds naturally for ε small enough.
On the other hand, it is easy to verify that the function

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \geqslant 0,$$

has the only minimal point $\bar{t} = \frac{N-p}{p}$ and is increasing on the interval $(\bar{t}, +\infty)$. Thus for $N \geqslant p^2$ we deduce that

$$\frac{N}{p} \leqslant b(\mu) \quad \Longleftrightarrow \quad f\left(\frac{N}{p}\right) \leqslant f\big(b(\mu)\big) = 0 \quad \Longleftrightarrow \quad 0 \leqslant \mu \leqslant \frac{N-p^2}{p}\left(\frac{N}{p}\right)^{p-1}.$$

Thus under the assumptions of case (ii), (2.37) holds for ε small.

(iii) $1 < q < p$.

If $1 < q < \min\{p, \frac{N}{b(\mu)}\}$, then we have that

$$\int_{\mathbb{R}^N} |v_\varepsilon|^q \geqslant C\varepsilon^{q(b(\mu)+1-\frac{N}{p})}$$

and

$$q\left(b(\mu) + 1 - \frac{N}{p}\right) < b(\mu)p + p - N,$$

which together with (2.40) imply that (2.37) holds for ε small.

If $\tilde{q} < q < p$ with

$$\tilde{q} = \max\left\{1, \frac{N}{b(\mu)}, \frac{p(2N - p - pb(\mu))}{N - p}\right\},$$

then we have that

$$\int_{\mathbb{R}^N} |v_\varepsilon|^q \geqslant C\varepsilon^{N+q(1-\frac{N}{p})},$$

and

$$N + q\left(1 - \frac{N}{p}\right) < b(\mu)p + p - N.$$

By taking ε small enough, from (2.40) we also get (2.37).

Combining (i)–(iii), we complete the proof of the lemma. □

3. Proof of the main result

In this section, we will give the proofs of Theorems 1.1–1.3.

Proof of Theorems 1.1 and 1.2. From Lemmas 2.5 and 2.6, we can apply Lemma 2.2 to the functional I_n to get a positive level $c_{\lambda,n}$ and a $(PS)_{c_{\lambda,n}}$ sequence $\{u_j^{(n)}\}_j \subset D^{1,p}(\mathbb{R}^N)$, such that

$$I_n\big(u_j^{(n)}\big) \to c_{\lambda,n} \quad \text{and} \quad I_n'\big(u_j^{(n)}\big) \to 0 \quad \text{as } j \to \infty.$$

Furthermore, Lemmas 2.5 and 2.7 imply that

$$0 < \beta \leqslant c_{\lambda,n} = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} I_n(u) \leqslant d_\lambda < \frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}}.$$

Passing to a subsequence if necessary, we can get some $c_\lambda \in [\beta, d_\lambda]$ such that

$$c_\lambda = \lim_{n\to\infty} c_{\lambda,n},$$

then for all $\varepsilon > 0$ small, there exists $n_0 > 0$ such that

$$c_{\lambda,n} \in (c_\lambda - \varepsilon, c_\lambda + \varepsilon), \quad \forall n > n_0.$$

Thus for every $n > n_0$, there exists $u_n = u_{j_n}^{(n)}$ satisfying

$$c_\lambda - \varepsilon < I_n(u_n) < c_\lambda + \varepsilon \quad \text{and} \quad \big\|I_n'(u_n)\big\| \leqslant \frac{1}{n}. \tag{3.1}$$

We claim that the sequence $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$.

In fact, for all $r > 0$ and n large enough, from (3.1) we get

$$I_n(u_n) - \frac{1}{r}\big\langle I_n'(u_n), u_n\big\rangle \leqslant C + \frac{1}{r}\|u_n\| \tag{3.2}$$

for some constant $C > 0$.

For $1 < q \leqslant p$ we can choose $m \in (p, p^*(s))$ such that

$$\begin{aligned}
I_n(u_n) - \frac{1}{m}\big\langle I_n'(u_n), u_n\big\rangle &= \left(\frac{1}{p} - \frac{1}{m}\right)\|u_n\|^p + \left(\frac{1}{m} - \frac{1}{p^*(s)}\right)\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s} - \lambda\left(\frac{1}{q} - \frac{1}{m}\right)\int_{\mathbb{R}^N} a_n(x)\big(u_n^+\big)^q \\
&\geqslant \left(\frac{1}{p} - \frac{1}{m}\right)\|u_n\|^p + \left(\frac{1}{m} - \frac{1}{p^*(s)}\right)\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s} - \lambda\left(\frac{1}{q} - \frac{1}{m}\right)\int_{\mathbb{R}^N} a_n^+(x)\big(u_n^+\big)^q \\
&\geqslant \left(\frac{1}{p} - \frac{1}{m}\right)\|u_n\|^p - \lambda\left(\frac{1}{q} - \frac{1}{m}\right)(A_{\mu,0})^{-\frac{q}{p}}\|u_n\|^q \big\|a^+(x)\big\|_{L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)}.
\end{aligned} \tag{3.3}$$

If $1 < q < p$, (3.2) and (3.3) imply that $\{u_n\}$ must be bounded in $D^{1,p}(\mathbb{R}^N)$.

If $q = p$, the same argument as that in Lemma 2.5 shows that there exist $C > 0$ and $\lambda^* > 0$ (defined as in Lemma 2.5), such that

$$I_n(u_n) - \frac{1}{p}\langle I_n'(u_n), u_n\rangle \geqslant C\|u_n\|^p \tag{3.4}$$

holds for all $\lambda \in (0, \lambda^*)$. Thus from (3.2) and (3.4), we also get the boundedness of $\{u_n\}$ in $D^{1,p}(\mathbb{R}^N)$.

Furthermore, for $q \in (p, p^*(s))$ we argue that

$$I_n(u_n) - \frac{1}{q}\langle I_n'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right)\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s} \geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p,$$

then (3.2) implies that $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$.

On the other hand, for $\lambda > 0$, $0 < s < p$, $p^*(s) \leqslant q < p^*$ and $a(x) \geqslant 0$ we have

$$I_n(u_n) - \frac{1}{p^*(s)}\langle I_n'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\|u_n\|^p + \lambda\left(\frac{1}{p^*(s)} - \frac{1}{q}\right)\int_{\mathbb{R}^N} a(x)\left(u_n^+\right)^q \geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p.$$

From (3.2) we get that $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$.

Now applying Lemma 2.4 to the sequence $\{u_n\}$, we can get a solution $u \in D^{1,p}(\mathbb{R}^N)$ for (1.1) satisfying $u \geqslant 0$ in $\mathbb{R}^N$. Next we verify that $u \not\equiv 0$.

Arguing by contradiction, we assume $u \equiv 0$. Setting

$$l = \lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s},$$

then from Lemma 2.3 we get

$$0 = \lim_{n\to\infty}\langle I_n'(u_n), u_n\rangle \geqslant \lim_{n\to\infty}\left(\|u_n\|^p - \int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s} - \lambda\int_{\mathbb{R}^N} a^+(x)\left(u_n^+\right)^q\right) \geqslant \lim_{n\to\infty}\|u_n\|^p - l,$$

that is,

$$\lim_{n\to\infty}\|u_n\|^p \leqslant l.$$

If $l = 0$, then we get $\lim_{n\to\infty} I_n(u_n) = 0$, which contradicts with (3.1). Thus we conclude that $l > 0$. Furthermore, Sobolev–Hardy inequality implies that

$$\|u_n\|^p \geqslant \left\|u_n^+\right\|^p \geqslant A_{\mu,s}\left(\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s}\right)^{\frac{p}{p^*(s)}}.$$

Then as $n \to \infty$ we have

$$l \geqslant \lim_{n\to\infty}\|u_n\|^p \geqslant A_{\mu,s}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} \frac{(u_n^+)^{p^*(s)}}{|x|^s}\right)^{\frac{p}{p^*(s)}} = A_{\mu,s}\, l^{\frac{p}{p^*(s)}}, \tag{3.5}$$

which implies that

$$l \geqslant (A_{\mu,s})^{\frac{N-s}{p-s}}. \tag{3.6}$$

Furthermore, form (3.1) it follows that

$$\frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}} > d_\lambda \geqslant c_\lambda = \lim_{n\to\infty} I_n(u_n). \tag{3.7}$$

From Lemma 2.3 we deduce that

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} a_n(x)\big(u_n^+\big)^q\right| \leqslant \lim_{n\to\infty}\int_{\mathbb{R}^N}\big(a^+(x)-a^-(x)\big)\big(u_n^+\big)^q = 0,$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a_n(x)\big(u_n^+\big)^q = 0. \tag{3.8}$$

Hence, for $q\in(1,p^*)$ and $m\in(p,p^*(s))$, from (3.5)–(3.8) we get

$$\begin{aligned}
\frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}} &> \lim_{n\to\infty} I_n(u_n) = \lim_{n\to\infty}\left(I_n(u_n) - \frac{1}{m}\big\langle I_n'(u_n), u_n\big\rangle\right)\\
&= \left(\frac{1}{p}-\frac{1}{m}\right)\lim_{n\to\infty}\|u_n\|^p + \left(\frac{1}{m}-\frac{1}{p^*(s)}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{(u_n^+)^{p^*(s)}}{|x|^s}\\
&\quad + \lambda\left(\frac{1}{m}-\frac{1}{q}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N} a_n(x)\big(u_n^+\big)^q\\
&> \left(\frac{1}{p}-\frac{1}{m}\right)(A_{\mu,s})l^{\frac{p}{p^*(s)}} + \left(\frac{1}{m}-\frac{1}{p^*(s)}\right)l\\
&\geqslant \left(\frac{1}{p}-\frac{1}{m}\right)(A_{\mu,s})^{1+\frac{N-p}{p-s}} + \left(\frac{1}{m}-\frac{1}{p^*(s)}\right)(A_{\mu,s})^{\frac{N-s}{p-s}}\\
&= \frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}}.
\end{aligned}$$

Therefore we get a contradiction, which implies that $u\not\equiv 0$.

Thus the proof of Theorems 1.1 and 1.2 is completed. □

Proof of Theorem 1.3. Now we employ the extremal function V_ε defined as in Lemma 2.1. To this end, we choose $n_0\in\mathbb{N}$ with $B(0,2R)\subset B(0,n_0)$, where R is defined as in Lemma 2.5. For $n>n_0$ and $t\geqslant 0$ we have

$$\begin{aligned}
I_n(tV_\varepsilon) &= \left(\frac{t^p}{p}-\frac{t^{p^*(s)}}{p^*(s)}\right)(A_{\mu,s})^{\frac{N-s}{p-s}} - \lambda\frac{t^q}{q}\int_{\mathbb{R}^N}\phi_n(x)a(x)|V_\varepsilon|^q\\
&\leqslant \left(\frac{t^p}{p}-\frac{t^{p^*(s)}}{p^*(s)}\right)(A_{\mu,s})^{\frac{N-s}{p-s}} - \lambda\bar{a}_0\frac{t^q}{q}\int_{B(0,2R)}|V_\varepsilon|^q,
\end{aligned}$$

where $\bar{a}_0>0$ is the minimal value of $a(x)$ on $\bar{B}(0,2R)$. Then the following functional $J_\lambda(tV_\varepsilon)$ is well defined

$$J_\lambda(tV_\varepsilon) := \left(\frac{t^p}{p}-\frac{t^{p^*(s)}}{p^*(s)}\right)(A_{\mu,s})^{\frac{N-s}{p-s}} - \lambda\bar{a}_0\frac{t^q}{q}\int_{B(0,2R)}|V_\varepsilon|^q.$$

The sequence V_ε^q is bounded in $L^{\frac{p^*}{q}}(\mathbb{R}^N)$ and $V_\varepsilon(x)\to 0$ a.e. in $\mathbb{R}^N$ as $\varepsilon\to 0$, thus $V_\varepsilon\to 0$ weakly in $L^{\frac{p^*}{q}}(\mathbb{R}^N)$ as $\varepsilon\to 0$. The restriction $V_\varepsilon|_{B(0,2R)}$ belongs to $W^{1,p}(B(0,2R))$, by the Sobolev embedding theorem, $V_\varepsilon\to 0$ strongly in $L^q(B(0,2R))$ for every $1\leqslant q<p^*$, therefore

$$\lim_{\varepsilon\to 0}\int_{B(0,2R)}|V_\varepsilon|^q = 0$$

and there exists $\varepsilon_0>0$ such that

$$0 < J_\lambda(V_{\varepsilon_0}) < \frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}}. \tag{3.9}$$

Arguing as in the proof of Lemma 2.6, we can take $t_{\varepsilon_0} > 0$ so that

$$J_\lambda(t_{\varepsilon_0} V_{\varepsilon_0}) = \sup_{t \geqslant 0}\left\{J_\lambda(t V_{\varepsilon_0})\right\} \quad \text{and} \quad \frac{d}{dt}\bigg|_{t=t_{\varepsilon_0}} J_\lambda(t V_{\varepsilon_0}) = 0,$$

that is,

$$\left(t_{\varepsilon_0}^{p-1} - t_{\varepsilon_0}^{p^*(s)-1}\right)(A_{\mu,s})^{\frac{N-s}{p-s}} - \lambda \bar{a}_0 t_{\varepsilon_0}^{q-1} \int\limits_{B(0,2R)} |V_{\varepsilon_0}|^q = 0. \tag{3.10}$$

By (3.9) and (3.10) we obtain that $0 < t_{\varepsilon_0} < 1$. Furthermore, the fact that

$$\sup_{t \geqslant 0}\left(\frac{t^p}{p} - \frac{t^{p^*(s)}}{p^*(s)}\right) = \frac{p-s}{p(N-s)}$$

implies

$$J_\lambda(t_{\varepsilon_0} V_{\varepsilon_0}) \leqslant d_\lambda := J_\lambda(V_{\varepsilon_0}) < \frac{p-s}{p(N-s)}(A_{\mu,s})^{\frac{N-s}{p-s}}.$$

Then following the same arguments as in the proof of Theorems 1.1 and 1.2, we obtain the existence of nontrivial solutions to (1.1) under the assumptions of Theorem 1.3. By the strong maximum principle [23], these solutions are positive. Thus we complete the proof of Theorem 1.3. □

References

[1] B. Abdellaoui, V. Felli, I. Peral, Existence and nonexistence for quasilinear equations involving the p-Laplacian, Boll. Unione Mat. Ital. Sez. B 9 (2006) 445–484.

[2] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[3] A.K. Ben-Naoum, C. Troestler, M. Willem, Extrema problems with critical Sobolev exponents on unbounded domains, Nonlinear Anal. 26 (1996) 823–833.

[4] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequality with weights, Compos. Math. 53 (1984) 259–275.

[5] D. Cao, P. Han, Solutions for semilinear elliptic equations with critical exponents and Hardy potential, J. Differential Equations 205 (2004) 521–537.

[6] D. Cao, P. Han, Solutions to critical elliptic equations with multi-singular inverse square potentials, J. Differential Equations 224 (2006) 332–372.

[7] D. Cao, S. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev exponent and Hardy terms, J. Differential Equations 193 (2003) 424–434.

[8] A. Ferrero, F. Gazzola, Existence of solutions for singular critical growth semilinear elliptic equations, J. Differential Equations 177 (2001) 494–522.

[9] J.P. Garcia Azorero, I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998) 441–476.

[10] N. Ghoussoub, C. Yuan, Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents, Trans. Amer. Math. Soc. 352 (2000) 5703–5743.

[11] J.V. Goncalves, C.O. Alves, Existence of positive solutions for m-Laplacian equations in $\mathbb{R}^N$ involving critical Sobolev exponents, Nonlinear Anal. 32 (1998) 53–70.

[12] P. Han, Quasilinear elliptic problems with critical exponents and Hardy terms, Nonlinear Anal. 61 (2005) 735–758.

[13] G. Hardy, J. Littlewood, G. Polya, Inequalities, Cambridge University Press, Cambridge, UK, 1934.

[14] E. Jannelli, The role played by space dimension in elliptic critical problems, J. Differential Equations 156 (1999) 407–426.

[15] D. Kang, On the quasilinear elliptic problems with critical Sobolev–Hardy exponents and Hardy terms, Nonlinear Anal. (2007), doi:10.1016/j.na.2007.01.024.

[16] D. Kang, S. Peng, Positive solutions for elliptic equations with critical Sobolev–Hardy exponents, Appl. Math. Lett. 17 (2004) 411–416.

[17] D. Kang, S. Peng, Existence of solutions for elliptic equations with critical Sobolev–Hardy exponents, Nonlinear Anal. 56 (2004) 1151–1164.

[18] P.L. Lions, The concentration compactness principle in the calculus of variations, the limit case (I), Rev. Mat. Iberoamericana 1 (1) (1985) 145–201.

[19] P.L. Lions, The concentration compactness principle in the calculus of variations, the limit case (II), Rev. Mat. Iberoamericana 1 (2) (1985) 45–121.

[20] E. Silva, H. Soares, Quasilinear Dirichlet problems in $\mathbb{R}^N$ with critical growth, Nonlinear Anal. 43 (2001) 1–20.

[21] J. Simon, Regularité de la solution d'une equation non lineare dans $\mathbb{R}^N$, in: P. Benilan (Ed.), Lectures Notes in Math., vol. 665, Springer, Berlin, 1978.

[22] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.

[23] J.L. Vazquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.